

NONCOMPLEX SMOOTH 4-MANIFOLDS WITH GENUS-2 LEFSCHETZ FIBRATIONS

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ABSTRACT. We construct noncomplex smooth 4-manifolds which admit genus-2 Lefschetz fibrations over S^2 . The fibrations are necessarily hyperelliptic, and the resulting 4-manifolds are not even homotopy equivalent to complex surfaces. Furthermore, these examples show that fiber sums of holomorphic Lefschetz fibrations do not necessarily admit complex structures.

In the following we will prove the following theorem.

Theorem 1. *There are infinitely many (pairwise nonhomeomorphic) 4-manifolds which admit genus-2 Lefschetz fibrations but do not carry complex structure with either orientation.*

Matsumoto [M] showed that $S^2 \times T^2 \# 4\overline{\mathbb{CP}^2}$ admits a genus-2 Lefschetz fibration over S^2 with global monodromy $(\beta_1, \dots, \beta_4)^2$, where β_1, \dots, β_4 are the curves indicated by Figure 1. (For definitions and details regarding Lefschetz fibrations see [M], [GS].)

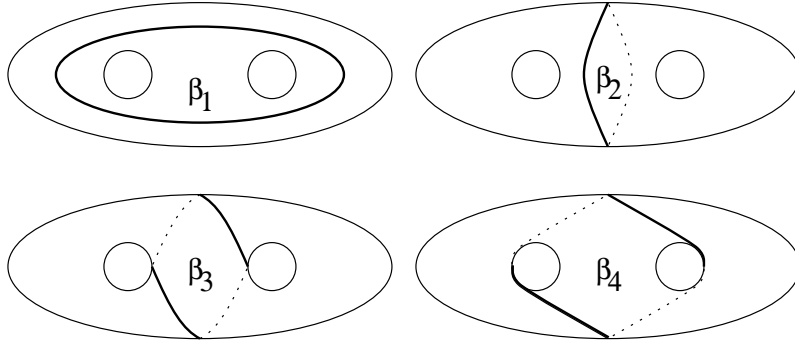


FIGURE 1.

Let B_n denote the smooth 4-manifold which admits a genus-2 Lefschetz fibration over S^2 with global monodromy

$$((\beta_1, \dots, \beta_4)^2, (h^n(\beta_1), \dots, h^n(\beta_4))^2)$$

where $h = D(a_2)$ is a positive Dehn twist about the curve a_2 indicated in Figure 2.

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Theorem 2. *For the 4-manifold B_n given above we have $\pi_1(B_n) = \mathbb{Z} \oplus \mathbb{Z}_n$.*

Proof. Standard theory of Lefschetz fibrations gives that

$$\pi_1(B_n) = \pi_1(\Sigma_2) / \langle \beta_1, \dots, \beta_4, h^n(\beta_1), \dots, h^n(\beta_4) \rangle.$$

Let $\{a_1, b_1, a_2, b_2\}$ be the standard generators for $\pi_1(\Sigma_2)$ (Figure 2).

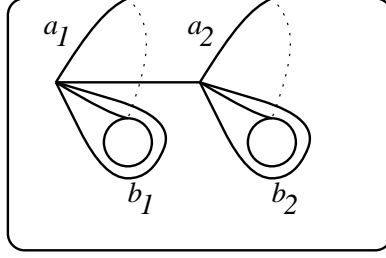


FIGURE 2.

Then we observe that

$$\beta_1 = b_1 b_2,$$

$$\beta_2 = a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1},$$

$$\beta_3 = b_2 a_2 b_2^{-1} a_1,$$

$$\beta_4 = b_2 a_2 a_1 b_1,$$

$$h^n(\beta_1) = b_1 b_2 a_2^n,$$

$$h^n(\beta_2) = \beta_2,$$

$$h^n(\beta_3) = \beta_3,$$

$$h^n(\beta_4) = b_2 a_2^{n+1} a_1 b_1.$$

Hence

$$\begin{aligned} \pi_1(B_n) = & \langle a_1, b_1, a_2, b_2 \mid b_1 b_2, [a_1, b_1], [a_2, b_2], b_2 a_2 b_2^{-1} a_1, b_2 a_2 a_1 b_1, b_1 b_2 a_2^n, b_2 a_2^{n+1} a_1 b_1 \rangle \\ = & \langle a_2, b_2 \mid [a_2, b_2], a_2^n \rangle = \mathbb{Z} \oplus \mathbb{Z}_n, \text{ and this concludes the proof.} \end{aligned}$$

□

Theorem 3. *B_n does not admit a complex structure.*

Proof. Assume that B_n admits a complex structure. Let M_n denote its n -fold cover for which $\pi_1(M_n) \cong \mathbb{Z}$ and M'_n the minimal model of M_n . By the theorem of Gompf [GS] B_n admits a symplectic structure, hence so does M_n and (by combining results of Taubes and Gompf [T], [G2]) M'_n . Consequently, if B_n is a complex surface, then we have a symplectic, minimal complex surface M'_n with $\pi_1(M'_n) \cong \mathbb{Z}$. In the following we will show that this leads to a contradiction.

By the Enriques-Kodaira classification of complex surfaces [BPV], (since $b_1(M'_n) = 1$) M'_n is either a surface of class *VII* (in which case $b_2^+(M'_n) = 0$), a secondary Kodaira surface (in which case $b_2(M'_n) = 0$) or a (minimal) properly elliptic surface.

Since M'_n is a symplectic 4-manifold, $b_2^+(M'_n)$ (and so $b_2(M'_n)$) is positive; this observation excludes the first two possibilities.

Suppose now that M'_n admits an elliptic fibration over a Riemann surface. If the Euler characteristic of M'_n is 0, then (following from the fact that $b_1(M'_n) = b_3(M'_n) = 1$) we get that $b_2(M'_n) = 0$, which leads to the above contradiction. Suppose finally that M'_n is a minimal elliptic surface with positive Euler characteristic. Since $b_1(M'_n) = 1$, it can only be fibered over S^2 (see for example [FM]). In that case (according to [G1], for example) its fundamental group is

$$\pi_1(M'_n) = \langle x_1, \dots, x_k \mid x_i^{p_i} = 1, i = 1, \dots, k; x_1 \cdots x_k = 1 \rangle.$$

This cannot be isomorphic to \mathbb{Z} , since if $\pi_1(M_n) \cong \mathbb{Z} = \langle a \rangle$, then $x_1 = a^{m_1}$ for some $m_1 \in \mathbb{Z}$, so a has finite order, which is a contradiction. Consequently the assumption that B_n is complex leads us to a contradiction, hence the theorem is proved. \square

Remark . The above proof, in fact, shows that B_n is not even homotopy equivalent to a complex surface — our arguments used only homotopic invariants (the fundamental group, b_2 and b_2^+) of the 4-manifold B_n . Note that basically the same idea shows that \overline{B}_n (the manifold B_n with the opposite orientation) carries no complex structure: The arguments involving the fundamental group, b_2 and the Euler characteristic only, apply without change. Since the fiber of the Lefschetz fibration on B_n is homotopically essential and provides a class with square 0, the intersection form of B_n and so of M_n are not definite — consequently these manifolds cannot be homotopy equivalent (with either orientation) to the blow-up of a surface of Class *VII*.

Proof of Theorem 1. By the definition of the 4-manifolds B_n we get infinitely many manifolds admitting genus-2 (consequently hyperelliptic) Lefschetz fibrations which are (by Theorem 2.) non-homeomorphic. As Theorem 3. and the above remark show, the manifolds B_n do not carry complex structures with either orientation, hence the proof of the Theorem 1. is complete. \square

Remark . We would like to point out that similar examples have been found by Fintushel and Stern [FS] — they used Seiberg-Witten theory to prove that their (simply connected) genus-2 Lefschetz fibrations are noncomplex.

Note that B_n is given as the fiber sum of two copies of $S^2 \times T^2 \# 4\overline{\mathbb{C}P^2}$, hence provides an example of the phenomenon that the fiber sum of holomorphic Lefschetz fibrations is not necessarily complex.

Acknowledgement. Examples of genus-2 Lefschetz fibrations with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_n$ were also constructed (as fiber sums) independently by Ivan Smith [S].

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